

# Reduction of Relativistic Three-Body Kinematics

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The Klein–Gordon system describing three scalar particles without interaction is cast into a new form by transformation of the momenta. Two redundant degrees of freedom are eliminated; we are left with a covariant equation for a reduced wave function with three-dimensional arguments. This new formulation of the mass-shell constraints is equivalent to the original KG system in a sector characterized by positivity of the energies and, if the mass differences are not too large, by a moderately relativistic regime. Introducing mutual interactions provides a model which is (at least for three equal masses) tractable and admits a reasonable nonrelativistic limit.

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**KEY WORDS:** relativistic wave equations; mass-shell constraints; three-body systems.

## 1. INTRODUCTION

### 1.1. Motivations

Relativistic *particle* dynamics is concerned with situations where the particles we consider are not significantly created or annihilated, whereas other relativistic effects must be taken into account. In principle the description of such particles should result from a specialization of quantum field theory (QFT) to its  $n$ -body sector.

This line leads to the famous integral equation of Bethe and Salpeter (BS) in the two-body case. Three-body generalizations were soon considered in the literature (Basdevant and Omnes, 1966; Pagnamenta, 1969a,b; Taylor, 1966). More recently (Bijtebier, 1999, 2000; Rupp and Tjon, 1992; Sammarruca *et al.*, 1992; Stadler and Gross, 1997). Four-body systems have been considered also (Bijtebier, 2001). For  $n > 2$  however, the complexity of the BS approach seems to be almost prohibitive as far as practical applications are concerned.

An alternative approach, based upon first principles (Bel, 1976, 1983; Droz-Vincent, 1975, 1979; Leutwyler and Stern, 1978a,b; Llosa, 1982; Todorov, 1976) uses  $n$  mass-shell constraints in the form of coupled wave equations where

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interaction terms can be either phenomenological or derived from QFT (Rizov *et al.*, 1985; Sazdjian, 1985, 1987).

This method shares with BS equation the property of manifest relativistic invariance, realized at the price of dealing with redundant degrees of freedom, since the arguments of the wave function are fourvectors. In the two-body case, there is a clue for eliminating the redundant degree of freedom: the sum of wave equations rules the dynamics, whereas their difference allows to determine how the wave function depends on the “relative time.” This dependence turns out to be trivial and one is left with a three-dimensional problem. In this framework, it is possible also to deal with fermions (Crater and Van Alstine, 1983, 1987; Crater *et al.*, 1992; Sazdjian, 1986).

In the three-body case we have to cope with two “relative times.” These superfluous degrees of freedom are present as well in the three-body versions of the BS equation. Their elimination (or factorization) is desirable for physical interpretation; it would produce (after diagonalization of the total linear momentum) a reduced wave equation which is covariant but similar to a Schroedinger equation with three-dimensional arguments. Unfortunately, the simple procedure utilized in the two-body case does not work for  $n > 2$ .

An important issue of  $n$ -body dynamics is cluster separability; but a less restrictive and more essential requirement is *global* separability: one must at least recover free-particle motion when *all* interactions are put equal to zero. Models violating global separability have been considered in the past (Droz-Vincent, 1985<sup>2</sup>; Iranzo *et al.*, 1981, 1984), mainly for their computational simplicity, but we believe that any reasonable formulation of  $n$ -body dynamics must include free motion as a limit when all the terms carrying interactions are “switched off.” Insofar as fermions are concerned, these matters have been discussed earlier in the literature (Bijtebier, 1990; Sazdjian, 1989).

For scalar particles with masses  $m_a$ , free motion can be described by  $n$  Klein–Gordon (KG) equations, say  $(p_a^2 - m_a^2)\Phi = 0$  where  $\Phi$  depends on the momenta  $p_1, \dots, p_n$ . We can give a sharp time-like value  $k^\alpha$  to the total linear momentum and use the differences of these equations. In the two-body case, it follows that the relative time (or alternatively the relative energy  $\frac{c}{2}(p_1 - p_2) \cdot k/\sqrt{k^2}$ , which is conjugate to it) arises only in a trivial factor of the wave function.

But this procedure is unable to produce any simplification as soon as  $n > 2$ . So we face this difficulty that *even for free particles*, the usual form of the equations of motion fails to permit the elimination of superfluous degrees of freedom.

This point may seem to be academic, because a system of noninteracting particles has no bound state, which renders a three-dimensional formulation unnecessary. But we bear in mind the eventuality of introducing interactions that

<sup>2</sup> In spite of the presence of  $D$ , the 3-body wave equation proposed therein cannot be identified with Eq. (104) of the present work. This remark also applies to Iranzo *et al.* (1981, 1984).

ultimately give rise to bound states. Therefore the possibility of a reduction is essential and *should survive in the free case*.

In this paper we focus on three-body systems and we firstly consider the case of noninteracting particles. Let us stress that the free system is not considered on its own right, but rather as preliminary to the further introduction of mutual interactions.

Since the KG equations as they stand do not permit a factorization of the dependence on relative times, it is natural to transform these equations into an equivalent system such that two superfluous degrees of freedom can be desentangled from the kinematics.

An early attempt to carry out this task for an arbitrary number of particles was made by Sazdjian several years ago (Sazdjian, 1988, 1989). Here, however, we shall be concerned with the three-body case only, and shall take advantage of a simplification which is not possible for  $n > 3$ . Our aim is to eliminate two degrees of freedom in such a way that the mass-shell constraints reduce to a covariant problem with three-dimensional arguments. Ultimate introduction of interactions will be briefly sketched at the end. Of course, the Poincaré invariance of kinematics must be preserved and all particles should be treated on equal footing (democracy). These conditions are not likely to select a unique scheme, but if we intend to make it as simple as possible, there are not too many choices.

We perform a rearrangement of the individual coordinates (well known in celestial mechanics) which is adapted to the consideration of relative variables. We insist on having invertible formulas, which is necessary to make sure that the new form of the equations of motion is equivalent to the original KG system.

Section 2 is devoted to an exposition of the notation used and of the basic useful equations of relativistic dynamics. In Section 3 we collect known results and perform elementary manipulations.

In Section 4, using the “heliocentric variables,” we construct in closed form a transformation of the free-particle system and discuss under which conditions this transformation is invertible. In Section 5, we briefly indicate how mutual interaction could be introduced.

## 2. BASIC EQUATIONS, NOTATION

Units are such that  $\hbar = 1$  whereas  $c$  remains unspecified. We start from the KG-system describing  $n$  particles in momentum representation

$$p_a^2 \Phi = m_a^2 c^2 \Phi, \quad a, b, c = 1 \dots n \quad (1)$$

where  $\Phi$  depends on the three four-vectors  $p_a^\alpha$ . Configuration and momentum variables are mutually conjugate:  $[q_a^\alpha, p_{b\beta}] = i \delta_{ab} \delta_\beta^\alpha$ , and so on. We make use of

the following notation:

$$Q = \frac{1}{n} \sum q_a, \quad P = \sum p_a, \quad z_{ab} = q_a - q_b \quad (2)$$

$$y_a = \frac{P}{n} - p_a \quad (3)$$

Moreover it is convenient to define

$$P_{ab} = p_a + p_b, \quad y_{ab} = \frac{1}{2}(p_a - p_b) \quad (4)$$

Beware that  $z_{ab}$  is *not* conjugate to  $y_{ab}$ . We obviously have the following relations:

$$y_a - y_b = -(p_a - p_b) = -2y_{ab} \quad (5)$$

$$\sum y_a = 0, \quad 2y_{ab} + y_a - y_b = 0, \quad \tilde{p}_a = -\tilde{y}_a$$

The tilde symbol denotes projection orthogonal to  $P^\alpha$ , in other words  $\tilde{y}_a = \prod y_a$ ,  $\tilde{z}_a = \prod z_a$ , with  $\prod = \delta - (P \otimes P)/P^2$ . Similarly, the “hat” symbol refers to the projection orthogonal to  $k^\alpha$ , eigenvalue of  $P^\alpha$ . For instance

$$\tilde{y}_a^\sigma = y_a^\sigma - (y_a \cdot P/P^2)P^\sigma$$

$$\hat{y}_a^\sigma = y_a^\sigma - (y_a \cdot k/k^2)k^\sigma$$

*Heliocentric variables.* The problem of “relative times” cannot be easily handled unless we first choose a set of independent *relative variables*. For this end, one particle is arbitrarily picked up; let it be particle with label 1. With respect to particle 1, the relative configuration variables are defined as in Iranzo *et al.* (1981):

$$z_A = q_1 - q_A \quad (6)$$

where the capital labels  $A, B, C$  run only from 2 to  $n$ . From (3) it follows that  $z_A$  is conjugate to  $y_A$ .

Let us now specialize to three-body systems; we can write

$$y_{12} = y_2 + \frac{1}{2}y_3, \quad y_{13} = y_3 + \frac{1}{2}y_2 \quad (7)$$

$$z_{12} + z_{23} + z_{31} = 0 \quad (8)$$

Notice that Eqs. (1–5) hold true for any  $n$ , whereas (7) and (8) are valid for  $n = 3$  only. It is clear that  $Q, z_2, z_3$  are independent configuration variables. In the same way  $P, y_2, y_3$  are independent momentum variables, canonically conjugate to them. We can use the set of canonical variables  $Q, z_2, z_3, P, y_2, y_3$  in place of  $q_1, q_2, q_3, p_1, p_2, p_3$ , this change is trivial. In this “heliocentric” formulation, democracy among the three particles is of course not kept manifest but can be checked at various stages of the development. A similar rearrangement, showing two relative momenta, is of current use in (Newtonian) celestial mechanics.

Among the quantities  $P_{ab}$  we shall more specially need to evaluate  $P_{12}$ ,  $P_{13}$ . They are given by

$$P_{12} = \frac{2}{3}P + y_3, \quad P_{13} = \frac{2}{3}P + y_2 \quad (9)$$

We shall also need the canonical expression of  $y_{12}$ ,  $y_{13}$ , given by (7). It will be convenient to replace Eqs. (1) by their sum and their differences; to this end we define

$$v_A = \frac{1}{2}(m_1^2 - m_A^2)$$

so the equal-mass case is characterized by the vanishing of both  $v_1$ ,  $v_2$ .

### 3. EQUATIONS OF MOTION

Equation (1) can obviously be written as

$$c^2 \sum m_a^2 \Phi = \sum p_a^2 \Phi \quad (10)$$

$$(m_a^2 - m_b^2)c^2 \Phi = (p_a^2 - p_b^2)\Phi \quad (11)$$

Notice that, according to notation (4),

$$\frac{1}{2}(p_a^2 - p_b^2) = y_{ab} \cdot P_{ab} \quad (12)$$

In Eq. (10), let us use the identity

$$n \sum_1^n p^2 \equiv P^2 + \sum_{a<b} (p_a - p_b)^2 \quad (13)$$

valid for any sum of n squares in a commutative algebra. We obtain

$$3 \sum m^2 c^2 \Phi = P^2 \Phi + \sum_{a<b} (p_a - p_b)^2 \Phi \quad (14)$$

In terms of the relative variables (see Eq. (7) we have another identity *specific of the three-body problem*

$$\sum_{a<b} (p_a - p_b)^2 \equiv 6(y_2^2 + y_3^2 + y_2 \cdot y_3) \quad (15)$$

Now in the r.h.s. of (15) we separate time from space according to the direction of  $P$ , and insert the result into (14). We get

$$\sum_{a<b} (p_a - p_b)^2 = D + 6P^2 \Xi \quad (16)$$

where

$$D = 6(\tilde{y}_2^2 + \tilde{y}_3^2 + \tilde{y}_2 \cdot \tilde{y}_3) \tag{17}$$

$$\Xi = (P^2)^{-2}[(y_2 \cdot P)^2 + (y_3 \cdot P)^2 + (y_2 \cdot P)(y_3 \cdot P)] \tag{18}$$

Thus the sum of Eq. (1) is

$$\left(3 \sum m^2 c^2 - P^2\right)\Phi = (D + 6P^2\Xi)\Phi \tag{19}$$

The remaining combinations of (1) can easily be written as the “difference equations”

$$y_{12} \cdot P_{12}\Phi = \nu_2 c^2 \Phi, \quad y_{13} \cdot P_{13}\Phi = \nu_3 c^2 \Phi \tag{20}$$

Now it is natural to require that  $\Phi$  is also eigenstate of the total momentum, say

$$P^\alpha \Phi = k^\alpha \Phi \tag{21}$$

for some time-like constant vector  $k$ . But (in contrast to what happens in the two-body case) this procedure is unable of getting rid of the relative energies  $c y_A \cdot k / \sqrt{k^2}$  conjugate to the relative times  $c^{-1} z_A \cdot k / \sqrt{k^2}$ .

Nevertheless, we can look for a new set of canonical variables; if these variables are suitably chosen, Eq. (20) may after all result in the elimination of two degrees of freedom.

#### 4. ALTERNATIVE FORMULATION OF THE FREE MOTION

##### 4.1. Transformations in Momentum Space

We shall construct a new representation of the KG system. It will involve a new set of operators  $q'_a, p'_b$  satisfying the canonical commutation relations. Let them be rearranged as  $P', z'_A, y'_B$  by formulas similar to (2), (3), and (6). In particular  $\sum y'_a$  vanishes and  $P' = \sum p'_a$  but we must require that  $P' = P$  to preserve translation invariance. Thus

$$y'_a = \frac{P}{n} - p'_a, \quad y'_{ab} = \frac{1}{2}(p'_a - p'_b)$$

$$y'_a - y'_b = -(p'_a - p'_b) = -2y'_{ab}$$

Naturally  $\sum y'_a \equiv 0$ . Notice for  $\sum_1^3 p'^2$  and for  $\sum_{a<b} (p'_a - p'_b)^2$  identities similar to (13) and (15). We obtain

$$y'_{12} = y'_2 + \frac{1}{2}y'_3 \quad y'_{13} = y'_3 + \frac{1}{2}y'_2 \tag{22}$$

$$y'_2 = \frac{4}{3}y'_{12} - \frac{2}{3}y'_{13} \quad y'_3 = \frac{4}{3}y'_{13} - \frac{2}{3}y'_{12} \tag{23}$$

Define

$$Q' = \frac{1}{3} \sum q'_a \quad z'_A = q'_1 - q'_A \quad (24)$$

It is clear that  $Q'$ ,  $z'_2$ ,  $z'_3$ ,  $P$ ,  $y'_2$ ,  $y'_3$  must be independent variables,  $y'_A$  conjugate to  $z'_A$ , etc.

A transformation in momentum space will be enough to induce the suitable transformation among operators. In fact we are going to construct the quantum analog of a *point transformation* in momentum space (see Appendix A).

Let us start with a wave function  $\Phi(p_1, p_2, p_3)$ . Perform a change in the space of its arguments

$$p_a \mapsto p'_b$$

or equivalently  $P \mapsto P' = P$  and  $y_A \mapsto y'_A$ . Instead of the old configuration variables  $z = i \frac{\partial}{\partial y}$ ,  $Q = i \frac{\partial}{\partial P}$ , we shall now consider

$$z' = i \frac{\partial}{\partial y'}, \quad Q' = i \frac{\partial}{\partial P'}$$

Since  $\partial P / \partial y' = 0$  and  $\partial P / \partial P' = \delta$ , the transformation formulas are as follows:

$$z'^{\alpha} = \frac{\partial y_B^{\sigma}}{\partial y'_{A\alpha}} z_{B\sigma} \quad (25)$$

$$Q'^{\alpha} = Q^{\alpha} + \frac{\partial y_A^{\mu}}{\partial P'_{\alpha}} z_{A\mu} \quad (26)$$

with summation also over (repeated) capital indices. In these formulas it is clear that the transformation of momenta must be invertible. Beware that  $Q'$  may not coincide with  $Q$  because of  $\partial y / \partial P'$ . In addition, we observe that the new relative coordinates actually mix the old ones. However, we shall prove later (Section 5) that this difficulty disappears in the large-total-mass limit.

It is in order to stress that finding the desired transformation amounts to solve a problem in the framework of  $c$ -numbers. The question of inverting formulas, discussed below, is nothing but a nonlinear problem concerning the arguments of the wave function. Since it is specified that we are dealing with momentum representation, we shall use without confusion the same symbols for the arguments of the wave function and the *multiplicative operators* they define.

For a better understanding of the mathematical structure, it is perhaps relevant to notice that  $Q'$  and  $z'_A$  are “formally hermitian” in this sense that they are symmetric operators in

$$\mathcal{L}'^2(R^{12}) = L^2(R^{12}, d^4 P d^4 y'_2 d^4 y'_3)$$

whereas  $Q$  and  $z_B$  are symmetric operators in

$$\mathcal{L}^2(R^{12}) = L^2(R^{12}, d^4 P d^4 y_2 d^4 y_3)$$

In contrast the momenta are symmetric operators in both senses. For mathematical convenience we shall work with a new wave function  $\Psi = |J|^{1/2}\Phi$ , where  $J$  is the Jacobian  $J = \frac{D(p_1, p_2, p_3)}{D(p'_1, p'_2, p'_3)}$  always finite and nonvanishing insofar as our transformation is invertible. Indeed multiplication by  $|J|^{1/2}$  maps  $\mathcal{L}^2$  onto  $\mathcal{L}'^2$ , so  $\Phi$  (resp.  $\Psi$ ) belongs to the rigged-Hilbert space constructed by taking  $\mathcal{L}^2$  (resp.  $\mathcal{L}'^2$ ) as Hilbert space. Although  $\mathcal{L}^2$  (resp.  $\mathcal{L}'^2$ ) has no direct physical meaning, it allows for representing the Poincaré algebra and gives a rigorous status to the operators involved in the wave equations.

Since  $p_a$  are multiplicative operators, they commute with  $J$ , so the mass shell constraints can be written either as (1) for  $\Phi$  or equivalently in the form  $(p_a^2 - m_a^2 c^2)\Psi = 0$ , with each  $p_a$  expressed in terms of  $p'_b$ . In momentum space, the Lorentz group is characterized by this property that it leaves all the products  $p_a \cdot p_b$  unchanged.

Provided all the  $p'_a \cdot p'_b$  can be expressed as functions of  $p_c \cdot p_d$  and vice versa, the same realization of the Lorentz group can be as well characterized by invariance of all the scalar products  $p_c \cdot p_d$ . In such a situation, although  $M' = \sum q' \wedge p'$  may be distinct from  $M = \sum q \wedge p$ , their components span the same Lie algebra. Moreover  $J$  being conserved by rotations, it follows that  $M$  and  $M'$  are both symmetric in  $\mathcal{L}^2$  and also in  $\mathcal{L}'^2$ .

Till now we have considered a large class of transformations, characterized by Eqs. (25) and (26); the classical (nonquantum) limit of such formulas would define *point transformations in momentum space*.

We now specialize to a transformation which allows for eliminating the superfluous degrees of freedom. All we need is an invertible transformation such that

$$P_{12} \cdot y_{12} = P \cdot y'_{12}, \quad P_{13} \cdot y_{13} = P \cdot y'_{13} \tag{27}$$

Indeed, if these relations are satisfied, (20) takes on the form

$$y'_{1A} \cdot P\Psi = v_A c^2 \Psi \tag{28}$$

Then according to (23) the “difference equations” are

$$y'_2 \cdot P\Psi = \left(\frac{4}{3}v_2 - \frac{2}{3}v_3\right) c^2 \Psi \tag{29}$$

$$y'_3 \cdot P\Psi = \left(\frac{4}{3}v_3 - \frac{2}{3}v_2\right) c^2 \Psi \tag{30}$$

With help of Eq. (21) we obtain

$$\Psi = \delta(P^\alpha - k^\alpha) \delta\left(y'_2 \cdot k - \frac{4}{3}v_2 c^2 - \frac{2}{3}v_3 c^2\right) \delta\left(y'_3 \cdot k - \frac{4}{3}v_3 c^2 - \frac{2}{3}v_2 c^2\right) \psi \tag{31}$$

where  $\psi$  depends on  $y'_2, y'_3$  only through their orthogonal projections onto the three-plane orthogonal to  $k$ . One remains with the problem of determining a reduced (or *internal*) wave function  $\psi$ , which has no more arguments than the wave



function of a nonrelativistic problem. The dependence of the wave function on  $y'_A \cdot k$  is now factorized out.

For simplicity we complete our transformation law by imposing that the space projections of  $y_2, y_3$  (with respect to the rest frame) remain unchanged, say

$$\tilde{y}'_A = \tilde{y}_A \quad (32)$$

and just transform their time components in the way dictated by Eq. (27). This choice obviously preserves Lorentz invariance; we shall prove below that it does not destroy the democracy among particles.

In view of equation (32), and taking into account the identity

$$y'_{A\alpha} \equiv \frac{y'_A \cdot P}{P^2} P_\alpha + \tilde{y}'_{A\alpha} \quad (33)$$

it is clear that our change of variables is essentially determined by (27). As they stand, these formulas implicitly define  $y'_2 \cdot P$  and  $y'_3 \cdot P$  in terms of the old variables; but we still have to solve (27) for  $y'_2 \cdot P$  and  $y'_3 \cdot P$  in order to exhibit the transformation in closed form.

According to (9) and (7) the left-hand sides of conditions (27) are as follows:

$$P_{12} \cdot y_{12} = \frac{2}{3} y_2 \cdot P + y_2 \cdot y_3 + \frac{1}{3} y_3 \cdot P + \frac{1}{2} y_3^2 \quad (34)$$

$$P_{13} \cdot y_{13} = \frac{2}{3} y_3 \cdot P + y_2 \cdot y_3 + \frac{1}{3} y_2 \cdot P + \frac{1}{2} y_2^2 \quad (35)$$

For the right-hand sides, Eq. (22) yields

$$y'_{12} \cdot P = \left( y'_2 + \frac{1}{2} y'_3 \right) \cdot P \quad (36)$$

$$y'_{13} \cdot P = \left( y'_3 + \frac{1}{2} y'_2 \right) \cdot P \quad (37)$$

Therefore the requirement that (27) is satisfied can be expressed as the linear system

$$\frac{2}{3} y_2 \cdot P + y_2 \cdot y_3 + \frac{1}{3} y_3 \cdot P + \frac{1}{2} y_3^2 = \left( y'_2 + \frac{1}{2} y'_3 \right) \cdot P \quad (38)$$

$$\frac{2}{3} y_3 \cdot P + y_2 \cdot y_3 + \frac{1}{3} y_2 \cdot P + \frac{1}{2} y_2^2 = \left( y'_3 + \frac{1}{2} y'_2 \right) \cdot P \quad (39)$$

to be solved for  $y'_2 \cdot P$  and  $y'_3 \cdot P$ . The outcome of system (38) and (39) is

$$y'_2 \cdot P = \frac{2}{3} (y_2 \cdot P + y_2 \cdot y_3 + y_3^2) - \frac{1}{3} y_2^2 \quad (40)$$

$$y'_3 \cdot P = \frac{2}{3} (y_3 \cdot P + y_2 \cdot y_3 + y_2^2) - \frac{1}{3} y_3^2 \quad (41)$$

whereto we insert the decomposition (33). This substitution, together with (32), determines in closed form the transformation of momenta. But it remains to be checked that this transformation is invertible.

*Translation invariance* was ensured from the outset by assuming that  $P' = P$ .

*Lorentz invariance* is preserved because all the quadratic scalar quantities formed with the vectors  $P$ ,  $y'_2$ ,  $y'_3$  are scalar invariant under spacetime rotations.

*Democracy* between particles is not manifest in the heliocentric notation. Nevertheless it is not difficult to check that our way of transforming momentum variables treats all three particles on the same footing. Indeed we first observe that (32) entails  $\tilde{y}'_1 = \tilde{y}_1$ , which amounts to finally write  $\tilde{y}'_a = \tilde{y}_a$  for the three particles. Then using (12) and (4), we realize that (27) automatically implies a third relation  $P_{23} \cdot y_{23} = P \cdot y'_{23}$ .

## 4.2. Inversion of Formulas

Now that all components of the new momenta  $p'_a$  are determined we can (in principle) evaluate the configuration variables through formulas (25) and (26). It is essential to realize that our transformation of the momenta among themselves must be invertible: if it were not, the transformation would not be canonical and the new form given to the wave equations would not be equivalent with the KG system.

Formula (25) can be written in closed form provided we are able to carry out this inversion. We are thus faced with the problem of mapping the new momenta back onto the old ones, which amounts to solve the system (38) and (39) now for the unknown  $y_2 \cdot P$ ,  $y_3 \cdot P$  in terms of  $y'_2 \cdot P$ ,  $y'_3 \cdot P$ , assuming this time that the latter are given and taking (32) into account.

*Positive-energy condition.* The domain where (38) and (39) must be inverted can be limited to the *positive-energy sector*. So we require not only that  $P$  is time-like and future-oriented, but also that *every vector  $p_a$  is time-like and points toward the future*, which entails  $P \cdot p_a > 0$  and  $p_a \cdot p_b > 0$ .

At this stage it is convenient to introduce the dimensionless quantities

$$\xi = \frac{y_2 \cdot P}{P^2}, \quad \eta = \frac{y_3 \cdot P}{P^2} \quad (42)$$

thus (18) becomes

$$\Xi = \xi^2 + \eta^2 + \eta \cdot \xi \quad (43)$$

The positive-energy condition above implies limitations for  $\xi$  and  $\eta$ . Indeed we first derive from (3)

$$\xi = \frac{1}{3} - \frac{P \cdot p_2}{P^2}, \quad \eta = \frac{1}{3} - \frac{P \cdot p_3}{P^2} \quad (44)$$

From positivity of  $P \cdot p_A$  we get

$$\xi < \frac{1}{3}, \quad \eta < \frac{1}{3} \quad (45)$$

On the other hand we have  $P \cdot p_1 = P^2 - P \cdot p_2 - P \cdot p_3$ . According to (44) this identity reads  $P \cdot p_1/P^2 = \frac{1}{3} + \xi + \eta$  and this expression also must be positive. We end up with

$$-\frac{1}{3} < \xi + \eta \quad (46)$$

With these limitations in mind, we now turn to the inversion of system (38) and (39). In view of the identities

$$y_A \cdot y_B \equiv \frac{(y_A \cdot P)(y_B \cdot P)}{P^2} + \tilde{y}_A \cdot \tilde{y}_B$$

We can write

$$y_2 \cdot y_3 = P^2 \xi \eta + \tilde{y}_2 \cdot \tilde{y}_3$$

$$y_2^2 = P^2 \xi^2 + \tilde{y}_2^2, \quad y_3^2 = P^2 \eta^2 + \tilde{y}_3^2$$

Insert these formulas into (34) and (35), and write (27). We get

$$\frac{2}{3}\xi + \frac{1}{3}\eta + \xi\eta + \frac{\eta^2}{2} + \frac{\tilde{y}_2 \cdot \tilde{y}_3}{P^2} + \frac{1}{2} \frac{\tilde{y}_3^2}{P^2} = \frac{y'_{12} \cdot P}{P^2} \quad (47)$$

$$\frac{2}{3}\eta + \frac{1}{3}\xi + \xi\eta + \frac{\xi^2}{2} + \frac{\tilde{y}_2 \cdot \tilde{y}_3}{P^2} + \frac{1}{2} \frac{\tilde{y}_2^2}{P^2} = \frac{y'_{13} \cdot P}{P^2} \quad (48)$$

Because of (32) all quantities of the form  $\tilde{y}_A \cdot \tilde{y}_B$  are already known.

The above system (47) and (48) is quadratic in the unknown quantities  $\xi, \eta$ . Define dimensionless quantities  $u, v$  through the formulas

$$P^2 u = y'_{12} \cdot P - \left( \tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2} \tilde{y}_3^2 \right), \quad P^2 v = y'_{13} \cdot P - \left( \tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2} \tilde{y}_2^2 \right) \quad (49)$$

They are regarded as functions of the new momenta, since  $P$  and  $\tilde{y}_A$  coincide with  $P'$  and  $\tilde{y}'_A$ , respectively. Inserting (22) into (49) yields

$$P^2 u = y'_2 \cdot P + \frac{1}{2} y'_3 \cdot P - \tilde{y}'_2 \cdot \tilde{y}'_3 - \frac{1}{2} (\tilde{y}'_3)^2 \quad (50)$$

$$P^2 v = y'_3 \cdot P + \frac{1}{2} y'_2 \cdot P - \tilde{y}'_2 \cdot \tilde{y}'_3 - \frac{1}{2} (\tilde{y}'_2)^2 \quad (51)$$

The system (47) and (48) becomes

$$\frac{2}{3}\xi + \frac{1}{3}\eta + \xi\eta + \frac{\eta^2}{2} = u \quad (52)$$

$$\frac{2}{3}\eta + \frac{1}{3}\xi + \xi\eta + \frac{\xi^2}{2} = v \quad (53)$$

to be solved for  $\xi, \eta$  with  $u, v$  as in (49). Setting

$$3(u + v) = \sigma, \quad 3(u - v) = \epsilon$$

system (52) and (53) can be cast into the form

$$\xi + \eta + 2\eta\xi + \frac{1}{2}(\xi^2 + \eta^2) = \frac{\sigma}{3} \tag{54}$$

$$\frac{1}{3}\xi - \frac{1}{3}\eta + \frac{1}{2}(\eta^2 - \xi^2) = \frac{\epsilon}{3} \tag{55}$$

It is convenient to define  $X = \xi + \eta, Y = \xi - \eta$ . When inserted into (18), this change of variables produces

$$\Xi = \frac{3}{4}X^2 + \frac{1}{4}Y^2 \tag{56}$$

System (54) and (55) becomes

$$\frac{3}{4}X^2 + X - \frac{1}{4}Y^2 = \frac{\sigma}{3} \tag{57}$$

$$2Y - 3XY = 2\epsilon \tag{58}$$

The positive-energy conditions (45) and (46) demand that  $X$  belongs to the open interval  $(-\frac{1}{3}, \frac{2}{3})$  and also that  $Y > -1$ , which in turn require that  $X < \frac{2}{3}(1 + \epsilon)$ .

When  $\epsilon = 0$  a couple of obvious solutions is given by  $X = \frac{2}{3}$  (whatever is  $\sigma$ ) which corresponds to  $Y = \pm 2\sqrt{1 - \sigma/3}$ , but this possibility is ruled out by (45). Other solutions are given by  $Y = 0$  hence

$$X = X^\pm = \frac{2}{3}(-1 \pm \sqrt{1 + \sigma}) \tag{59}$$

but the solution  $X^-$  is excluded in view of condition (46).

We now turn to the general case. The possibility that strictly  $X = 2/3$  being discarded, we now solve (58)

$$Y = \frac{2\epsilon}{2 - 3X} \tag{60}$$

and bring the result into (57). Hence a fourth-degree polynomial equation to solve for  $X$ ,

$$(2 - 3X)^2 \left( \frac{3}{4}X^2 + X - \frac{\sigma}{3} \right) = \epsilon^2 \tag{61}$$

*Graphic analysis.* In principle such equation can be explicitly solved by radicals. But a graphic analysis gives a better understanding (see Fig. 1). Solving (61) amounts to discuss how, in the  $X, Z$  plane, the parametrized curves  $Z = R_\sigma(X) = (2 - 3X)^2(\frac{3}{4}X^2 + X - \frac{\sigma}{3})$  are intersected by a straight line with trivial equation  $Z = \epsilon^2$  (see Appendix B).

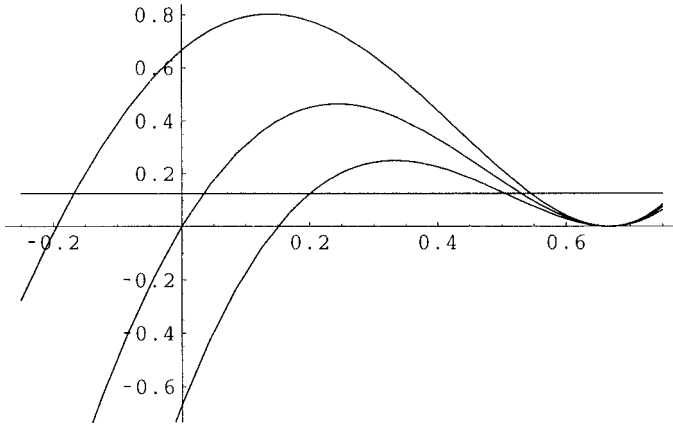


Fig. 1. Curves  $Z = R_\sigma(X)$  for  $\sigma = -1/2, \sigma = 0, \sigma = 1/2$ . The straight line  $Z = \epsilon^2$  is represented for  $\epsilon^2 = 1/8$ .

The outcome of graphic analysis is as follows.

**Proposition 4.I.** *Provided  $-\frac{3}{4} < \sigma < \frac{1}{2}$  and  $\epsilon$  is taken in the open interval  $(-\frac{1}{2}, \frac{1}{2})$ , among the real solutions of the system (57) and (58) there exists a unique one,  $X, Y$  such that  $X \in (-\frac{1}{3}, \frac{2}{3})$  and such that  $X$  reduces to  $X^+$  when  $\epsilon$  vanishes. Moreover we observe that  $X - \frac{2}{3}\epsilon$  remains bounded by  $\frac{2}{3}$ , ensuring that  $Y > -1$  as required among the positive-energy conditions.*

The expression  $X = S(\sigma, \epsilon)$  for this solution could be written in closed form, but is very complicated, except naturally for vanishing  $\epsilon$  where it is just given by  $X^+$ . For applications, we have better to use a development in powers of  $\epsilon^2$ , say

$$X = S(\sigma, \epsilon) = X^+ + \epsilon^2 X_{(1)} + \epsilon^4 X_{(2)} \cdots + \epsilon^{2p} X_{(p)} + \cdots \tag{62}$$

All coefficients  $X_{(p)}$  are derived from (61) and depend on  $\sigma$ . We find for instance  $X_{(1)} = \frac{4}{3(X^+ - X^-)(2 - 3X^+)^2}$ . Note that

$$S = \frac{\sigma}{3} + O(\sigma^2, \epsilon^2, \epsilon\sigma) \tag{63}$$

For the sake of a physical interpretation, investigating the behavior of our formulas at large  $P^2$  is of interest. Equations (50) and (51) show that, considered as functions of the independent variables  $y_A^\alpha, P^\beta$ , all the quantities  $u, v, \sigma, \epsilon, X, Y$  are of the order of  $1/|P|$ . We simply have

$$\xi = 2u - v + O(1/P^2), \quad \eta = 2v - u + O(1/P^2) \tag{64}$$

Proposition I stated above ensures that the transformation from the old momenta to the new ones is safely invertible in an open set of values given to the couple

$\sigma, \epsilon$ . As these quantities are first integrals for free particles, their limitation to an interval defines a sector which is invariant by the motion. Characterization of this sector in terms of physical quantities will be discussed in the next section.

*Remark.* Infinitely many other domains ensuring a unique solution to (57) and (58) could be exhibited. But we can enlarge the interval for  $\epsilon$  only at the price of shrinking the one for  $\sigma$ .

### 4.3. Physical Conditions

In view of Eq. (28), the wave function includes a factor  $\delta(y'_{12} \cdot P - c^2 v_2) \delta(y'_{13} \cdot P - c^2 v_3)$ . The relevant domain for the arguments of  $\Psi$  is thus limited by the constraints  $y'_{1A} \cdot P = c^2 v_A$ , where the masses are given from the outset. On the mass shell, we can replace  $y'_{1A} \cdot P$  by  $c^2 v_A$  in (49) or in the definitions of  $\sigma, \epsilon$ .

The particular case where  $\epsilon$  vanishes is interesting because it arises when the particles are mutually at rest, provided  $m_2 = m_3$ , which includes the special case where all masses are equal. Moreover  $\epsilon$  remains small insofar as  $v_2, v_3$  and the velocities are not too large.

For simplicity, let us focus on the assumption that  $v_2, v_3$  are small enough. To keep some contact with nonrelativistic mechanics, our scheme must encompass the case  $\epsilon = 0$ ; thus the solution which reduces to  $X^-$  for vanishing  $\epsilon$  is excluded. Since the transformation of momenta must be one-to-one, we are also obliged to discard the solutions which reduce to the fixed point for vanishing  $\epsilon$ .

Finally we have no other choice than the solution given by  $X = S(\sigma, \epsilon)$ .

Let us now discuss in more details how we can manage, by simple physical requirements, to keep  $\sigma, \epsilon$  within admissible values allowing to apply Proposition I.

From (49) we obtain

$$P^2 \frac{\sigma}{3} = (v_2 + v_3)c^2 - \left( 2\tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2}\tilde{y}_2^2 + \frac{1}{2}\tilde{y}_3^2 \right) \tag{65}$$

$$P^2 \frac{\epsilon}{3} = (v_2 - v_3)c^2 + \frac{1}{2}(\tilde{y}_2^2 - \tilde{y}_3^2) \tag{66}$$

in other words

$$\sigma = \sigma_0 - \frac{3}{P^2} \left( 2\tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2}\tilde{y}_2^2 + \frac{1}{2}\tilde{y}_3^2 \right) \tag{67}$$

$$\epsilon = \epsilon_0 + \frac{3}{2P^2} (\tilde{y}_2^2 - \tilde{y}_3^2) \tag{68}$$

setting

$$\sigma_0 = 3(v_2 + v_3)c^2/P^2, \quad \epsilon_0 = 3(v_2 - v_3)c^2/P^2 \tag{69}$$

In the domain where the arguments of  $\Psi$  vary, we can for instance impose a democratic condition

$$|\tilde{y}_a^2| < \frac{P^2}{24} \tag{70}$$

We remember that  $\tilde{p}_a = -\tilde{y}_a$ , thus condition (70) is a statement about individual momenta. From (70) it follows that

$$|\sigma| \leq |\sigma_0| + \frac{3}{P^2} \left| 2\tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2}\tilde{y}_2^2 + \frac{1}{2}\tilde{y}_3^2 \right| \tag{71}$$

$$|\epsilon| \leq |\epsilon_0| + \frac{3}{2P^2} |\tilde{y}_2^2 - \tilde{y}_3^2| \tag{72}$$

Since every  $\tilde{y}$  is space-like,  $|\tilde{y}_A \cdot \tilde{y}_B| \leq |\tilde{y}_A| |\tilde{y}_B|$ . Hence (70) implies

$$\begin{aligned} \left| 2\tilde{y}_2 \cdot \tilde{y}_3 + \frac{1}{2}\tilde{y}_2^2 + \frac{1}{2}\tilde{y}_3^2 \right| &\leq \frac{P^2}{8} \\ \left| \frac{1}{2}\tilde{y}_2^2 - \frac{1}{2}\tilde{y}_3^2 \right| &\leq \frac{P^2}{24} \end{aligned}$$

Therefore

$$|\sigma - \sigma_0| < \frac{3}{8} \tag{73}$$

$$|\epsilon - \epsilon_0| < \frac{1}{8} \tag{74}$$

Now, provided that

$$|\sigma_0| \leq \frac{1}{8}, \quad |\epsilon_0| \leq \frac{3}{8} \tag{75}$$

it stems from (73) and (74) that  $\sigma$  and  $\epsilon$  remain within the interval  $(-\frac{1}{2}, \frac{1}{2})$ . To realize this situation, we are led to restrict the squared-mass differences by the condition (75). Then, condition (70) permits to apply Proposition I.

Untill now, we have proposed condition (70), which involves not only the relative momenta but also  $P^2$ . Since we consider the positive-energy sector of free particles, it is clear that

$$P^2 > \sum p_a^2 = \sum m_a^2 c^2$$

For the sake of a simple kinematic interpretation, we have better to replace (70) by the stronger condition

$$|\tilde{p}_a^2| < \frac{1}{24} \sum m_a^2 c^2 \tag{76}$$

which is just a little more restrictive and offers the advantage of involving only masses and spatial velocities.

Similarly, in view of (69) it is clear that, to fulfill (75), it is sufficient to demand

$$|\nu_2 + \nu_3| < \sum \frac{m_a^2}{24}, \quad |\nu_2 - \nu_3| < \sum \frac{m_a^2}{8} \tag{77}$$

This approach is well-suited for the equal-mass case and remains useful when the mass differences are not too large.

*Example* (Two equal masses). Assume that  $m_A = \rho m_1$ , hence  $\sum m_a^2 = (1 + 2\rho^2)m_1^2$ . We find that (77) is satisfied provided the square-mass ratio satisfies  $\frac{23}{26} < \rho^2 < \frac{25}{22}$ . It is clear that (76) is a condition on the three-dimensional velocities with respect to the rest frame. Although it puts a bound on these quantities, it still leaves room for a large class of relativistic motions.

*Example* (Three equal masses). In the equal-mass case,  $m_a = m$ , thus both  $\nu_A$  vanish. We are sure that  $\sigma, \epsilon$  belong to the safety interval if we demand that

$$|\tilde{p}_a^2| < \frac{m^2 c^2}{8} \tag{78}$$

Indeed positivity entails that  $3m^2 c^2 \leq P^2$ .

Now what does mean (78) in terms of (Newtonian) velocities? In the rest frame, for all indices,  $|\tilde{p}^2| = m^2 \frac{\mathbf{w}^2}{1 - \mathbf{w}^2/c^2}$  where  $\mathbf{w}$  is the Newtonian velocity  $\frac{d\mathbf{x}}{dt}$ . Thus (78) is satisfied provided  $\mathbf{w}^2/c^2 < \frac{1}{9}$ , which corresponds to  $|\mathbf{w}| < c/3$ . Under this limit, say one third of the velocity of light, we shall speak of a “moderately relativistic regime.” For inequal masses, similar results could be derived, but the discussion would become a bit complicated. We summarize as follows.

**Proposition 4.II.** *Insofar as the mass differences are not too large, we keep the range of  $\sigma, \epsilon$  under control by restrictions on the magnitude of the velocities. If in particular we consider three equal masses, velocities under  $c/3$  ensure that we can invert our formulas with  $S(\sigma, \epsilon)$  as in Proposition I.*

All the quantities involved in condition (70) (resp. (76)) are first integrals for free particles, thus (70) (resp. (76)) defines an invariant sector of the motion.

**4.4. Individuality: New Versus Old Coordinates**

As a result of our transformation of the momenta, it might be puzzling that (besides its dependence on total momentum) each new variable  $q'_a$  depends not



only on  $q_a$  (with the same label  $a$ ) but also on all  $q_b$ 's with  $b \neq a$ . This dependence is expressed by the transformation formulas (25). Fortunately, we shall prove that

**Proposition 4.III.** *Besides its dependence on the direction of  $P$ , at zeroth order in  $1/|P|$ , the variable  $z'_2$  depends only on  $z_2$  (resp.  $z'_3$  depends only on  $z_3$ ).*

**Proof:** We develop our formulas in powers of  $1/|P|$  and evaluate  $z'_{A\alpha}$  at lowest order.

According to (25) we need to compute the coefficients  $\frac{\partial y}{\partial y'}$ .

Let us first prove that

$$\frac{\partial y_B^\sigma}{\partial y'_{A\alpha}} = O(1/|P|), \quad \text{for } A \neq B \quad (79)$$

From (42) and (32) it is clear that

$$y_2^\alpha = \tilde{y}_2^\alpha + \xi P^\alpha, \quad y_3^\alpha = \tilde{y}_3^\alpha + \eta P^\alpha \quad (80)$$

hence

$$\frac{\partial y_2^\sigma}{\partial y'_{3\alpha}} = \frac{\partial \xi}{\partial y'_{3\alpha}} P^\sigma \quad (81)$$

$$\frac{\partial y_2^\sigma}{\partial y'_{2\alpha}} = \Pi^{\sigma\alpha} + \frac{\partial \xi}{\partial y'_{2\alpha}} P^\sigma \quad (82)$$

and similar formulas for  $\partial y_3/\partial y'_A$ . We are led to evaluate the derivatives of  $\xi$  (resp.  $\eta$ ). According to (64) it is sufficient to differentiate  $u$  and  $v$ . With help of (50) and (51) we get

$$P^2 \frac{\partial u}{\partial y'_{2\alpha}} = P^\alpha - \tilde{y}_3^\alpha \quad (83)$$

$$P^2 \frac{\partial u}{\partial y'_{3\alpha}} = \frac{1}{2} P^\alpha - \tilde{y}_2^\alpha - \tilde{y}_3^\alpha \quad (84)$$

$$P^2 \frac{\partial v}{\partial y'_{2\alpha}} = \frac{1}{2} P^\alpha - \tilde{y}_2^\alpha - \tilde{y}_3^\alpha \quad (85)$$

$$P^2 \frac{\partial v}{\partial y'_{3\alpha}} = P^\alpha - \tilde{y}_2^\alpha \quad (86)$$

Let us insert (84) (86) and (83) (85) into the formulas obtained by differentiation of (64). We obtain

$$\frac{\partial \xi}{\partial y'_{3\alpha}} = O(1/P^2), \quad \frac{\partial \eta}{\partial y'_{2\alpha}} = O(1/P^2) \quad (87)$$

$$\frac{\partial \xi}{\partial y'_{2\alpha}} = \frac{3}{2} \frac{P^\alpha}{P^2} + O(1/P^2) \quad (88)$$

and a similar formula with  $\partial\eta/\partial y'_3$ . Inserting (87) into (81), we check that  $\frac{\partial y_2^\sigma}{\partial y'_{2\alpha}}$  actually is of the order of  $1/|P|$ , and the same result can be derived for  $\partial y_3/\partial y'_{2\alpha}$ , which altogether proves (79).

Now we apply formula (25) and take (79) into account. Hence

$$z_2'^\alpha = \frac{\partial y_2^\sigma}{\partial y'_{2\alpha}} z_{\sigma 2} + O(1/|P|) \tag{89}$$

But in view of (82) and (88) we simply have

$$\frac{\partial y_2^\sigma}{\partial y'_{2\alpha}} = \Pi^{\sigma\alpha} + \frac{3}{2} \frac{P^\sigma P^\alpha}{P^2} + O(1/|P|)$$

So finally

$$z_2'^\alpha = \tilde{z}_2^\alpha + \frac{3}{2} \frac{(z_2 \cdot P) P^\alpha}{P^2} + O(1/|P|) \tag{90}$$

and a similar expression in terms of  $z'_3, z_3$ . In particular we have

$$\tilde{z}'_2 = \tilde{z}_2 + O(1/|P|), \quad \tilde{z}'_3 = \tilde{z}_3 + O(1/|P|) \tag{91}$$

□

#### 4.5. New Form of Wave Equation

As seen in Section 4.1, the “difference equations” are (28) or equivalently (29) and (30). According to (19) the dynamical equation (sum equation) for free particles is

$$\left(3 \sum m_a^2 c^2 - P^2\right) \Psi = (D + 6P^2 \Xi) \Psi \tag{92}$$

Of course  $\Xi$  must be here considered as a function of  $y'_2 \cdot P, y'_3 \cdot P, P^2, \tilde{y}_A$ . In view of (56) and (60) we can write as well

$$\Xi = \frac{3}{4} X^2 + \frac{\epsilon^2}{(2 - 3X)^2} \tag{93}$$

where  $X = S(\sigma, \epsilon)$  according to (62) and (63). We must remember that  $\sigma, \epsilon$  are functions of the new momenta through (49).

But Eq. (28) tells that *on the mass shell* we can replace  $y'_{1A} \cdot P$  by  $v_A c^2$  (thus  $y'_A$  replaced accordingly, see Eqs. (29) and (30)). Moreover we impose that the total linear momentum has a sharp value  $k^\alpha$ . Let us make this convention that  $\underline{F}$  is the expression of any  $F$  on the momentum-mass shell, namely

$$\underline{F} = \text{subs}(y'_{1A} \cdot P = v_A c^2, P^\alpha = k^\alpha, F) \tag{94}$$

using an obvious notation borrowed from Maple's syntax. It is meant that  $y'_{1A}$  is as in (22) and we set

$$k^2 = M^2 c^2 \quad (95)$$

For instance, if we define

$$\hat{y}'_\alpha = y'_\alpha - (y' \cdot k/k^2)k_\alpha$$

we can write  $\hat{y}'^\alpha = \hat{y}^\alpha$ , therefore

$$\underline{D} = 6[(\hat{y}'_2)^2 + (\hat{y}'_3)^2 + \hat{y}'_2 \cdot \hat{y}'_3]$$

Moreover (49) yields

$$M^2 c^2 \underline{u} = v_2 c^2 - \left( \hat{y}_2 \cdot \hat{y}_3 + \frac{1}{2} \hat{y}_3^2 \right), \quad M^2 c^2 \underline{v} = v_3 c^2 - \left( \hat{y}_2 \cdot \hat{y}_3 + \frac{1}{2} \hat{y}_2^2 \right) \quad (96)$$

It is noteworthy that in the case of *two equal masses*  $\underline{\epsilon}$  is of the order of  $1/c^2$  whereas for *three equal masses* both  $\underline{\sigma}$  and  $\underline{\epsilon}$  are  $O(1/c^2)$ .

Taking into account the mass-shell constraints and the sharp value of  $P^\alpha$  we derive the reduced equation

$$\left( 3 \sum m_a^2 - M^2 \right) c^2 \psi = (\underline{D} + 6M^2 c^2 \underline{\Xi}) \psi \quad (97)$$

Notice that, apart from  $v_2, v_3$  that are fixed parameters,  $\underline{\Xi}$  depends only on  $\hat{y}'_2, \hat{y}'_3$ , and  $M^2$ . The only operators involved in (97) are multiplications by the projections of  $y'_A$  orthogonal to  $k$ , they are essentially three-dimensional. Whereas  $\underline{D}$  has a familiar form (just use the rest frame, where  $\hat{y}_A \cdot \hat{y}_B = -\mathbf{y}_A \cdot \mathbf{y}_B$ ), it is not the case for  $\underline{\Xi}$ . Fortunately it can be checked that, at least for equal masses, the term  $c^2 \underline{\Xi}$  is in fact of the order of  $1/c^2$ . For this purpose it is convenient to set

$$M^2 c^2 \underline{\Xi} = \frac{1}{M^2 c^2} \Gamma \quad (98)$$

so we end up with

$$\left( 3 \sum m_a^2 - M^2 \right) c^2 \psi = \underline{D} \psi + \frac{6}{M^2 c^2} \Gamma \psi \quad (99)$$

For *three equal masses*,  $\Gamma$  can be expanded in nonnegative powers of  $1/c^2$  and it turns out that its zeroth-order piece is biquadratic in  $\hat{y}'_A$ .

**Proof:** It can be easily read off from (49) that in this case  $\underline{u}, \underline{v}$  thus also  $\underline{\sigma}, \underline{\epsilon}$  are of the order of  $1/c^2$ . Getting back to system (52) and (53), one finds that

$$\underline{\xi} = 2\underline{u} - \underline{v} + O(1/c^4)$$

$$\underline{\eta} = 2\underline{v} - \underline{u} + O(1/c^4)$$

Inserting into (43) yields

$$\underline{\Xi} = 3(\underline{u}^2 + \underline{v}^2 - \underline{uv}) + O(1/c^4) \quad (100)$$

hence

$$M^4 c^4 \underline{\Xi} = \Gamma(0) + O(1/c^2) \quad (101)$$

$$\Gamma(0) = \frac{3}{4} [(\hat{y}_2^2)^2 + (\hat{y}_3^2)^2 + 4(\hat{y}_2 \cdot \hat{y}_3)^2 + 2(\hat{y}_2^2 + \hat{y}_3^2)(\hat{y}_2 \cdot \hat{y}_3) - (\hat{y}_2^2)(\hat{y}_3^2)] \quad (102)$$

Thus, when all  $m_a = m$ , the last term in the r.h.s. of (99) can be considered as small.  $\square$

Free-particle motion is now described only in terms of  $\hat{y}'$  and  $k$ .

Imposing by (21) that total linear momentum is diagonal permits, through Eq. (31), to eliminate  $y'_A \cdot k$ , where the new relative energies  $cy'_A \cdot k/\sqrt{k^2}$  are conjugate to the new “relative times.”

It is of interest to notice that these new “relative times” are linear combinations of the old ones with coefficients that are analytic functions of the momenta; the reader will check it, using (25)(26)(42) and (49)(36)(37).

After reduction, the three-body kinematics has no more degrees of freedom than in the nonrelativistic problem. But we must keep in mind that this picture is valid only insofar as we can revert to all the initial variables, which (at least for equal masses) is ensured for moderately relativistic velocities.

The new variables  $y'_A$  introduced in this section will be referred to as the *reducible variables*.

## 5. HOW TO INTRODUCE INTERACTIONS

We can now consider the system (92), (29), and (30) as a starting point for introducing mutual interactions.

To this end, we shall modify the “sum equation” (92) by a term which carries interaction, whereas the “difference equations” (29) and (30) remain untouched.

Doing so we manage that  $P$  remains conserved, and keep assuming that its eigenvalue is a time-like vector  $k$ ; therefore, the factorization of  $\Psi$  given by formula (31) remains valid and eliminates two degrees of freedom.

The interaction potential will be written in closed form in terms of the reducible coordinates  $z'_A, y'_B$ , and all calculations will be carried out using these variables.

*Remark :* The *reducible (momentum) coordinates*  $p'_a$  are rearranged as to form the quantities  $P$  and  $y'_A$ .

Adding interaction into (92) produces the dynamical equation

$$\left(3 \sum m_a^2 c^2 - P^2\right) \Psi = D\Psi + (18V + 6P^2 \Xi)\Psi \quad (103)$$

Like in the free case,  $D$  is given by (17) and  $\Xi$  is given by (93) in terms of  $X = S(\sigma, \epsilon)$ .

The “difference equations” remain (29) and (30) like previously. Of course,  $V$  cannot be chosen arbitrarily but it is not difficult to find a general admissible form of  $V$  such that the dynamical equation (103) is compatible with (29) and (30). Compatibility requires that  $V$  commutes with the operators in the left-hand sides of (29) and (30). For instance the interaction potential  $V$  may depend on  $\tilde{z}'_2, \tilde{z}'_3$ , and  $P^2$ .

Naturally  $V$  must be Poincaré invariant, which is realized by taking a function of the various scalar products formed with  $\tilde{z}'_A, \tilde{z}'_B, P$ . Demanding that  $\Psi$  diagonalizes  $P^\alpha$  with eigenvalue  $k^\alpha$ , with  $k \cdot k > 0$ , we can in (103) replace  $\tilde{y}'$  by  $\hat{y}'$ .

Taking (29), (30), and (98) into account yields the reduced equation

$$\left(3 \sum m_a^2 - M^2\right) c^2 \psi = \underline{D}\psi + 18\underline{V}\psi + \frac{6}{M^2 c^2} \Gamma \psi \quad (104)$$

where the reduced wave function  $\psi$  depends only on  $k$  and on the space projections  $\hat{y}'_2, \hat{y}'_3$ . The only operators involved here are the projections  $\hat{z}'_A, \hat{y}'_B$ . Moreover  $\hat{z}'$  arises in  $\underline{V}$  only.

Comparison with a standard problem of nonrelativistic quantum mechanics becomes more easy in the rest frame, where  $(\hat{z}'_A)^2 = -(\mathbf{z}'_A)^2$  and  $(\hat{y}'_A)^2 = -(\mathbf{y}'_A)^2$ , etc.

Actually solving (104) differs from a nonrelativistic problem by the last term, which involves the momenta but does not depend on the shape of the interaction (and survives in the free-motion limit). Still this term depends on the total squared mass.

For simplicity, we can consider an interaction such that

$$18V = \alpha_{12}U_{12}(\tilde{z}'_2) + \alpha_{23}U_{23}(\tilde{z}'_3 - \tilde{z}'_2) + \alpha_{31}U_{31}(\tilde{z}'_3) \quad (105)$$

where  $\alpha_{ab}$  are coupling constants and  $U_{ab}$  arbitrary (but Poincaré invariant) functions. In this model,  $U_{12}$  is independent from  $q'_3$ , etc., with cyclic permutation; the formal input of our interaction consists in two-body potentials.

So (104) can be written as

$$\begin{aligned} \left(3 \sum m_a^2 - M^2\right) c^2 \psi = \underline{D}\psi + [\alpha_{12}U_{12}(\hat{z}'_2) + \alpha_{23}U_{23}(\hat{z}'_3 - \hat{z}'_2) \\ + \alpha_{31}U_{31}(\hat{z}'_3)]\psi + \frac{6}{M^2 c^2} \Gamma \psi \end{aligned} \quad (106)$$

A special case

$$U_{12} = (\hat{z}'_2)^2, \quad U_{13} = (\hat{z}'_3)^2, \quad U_{23} = (\hat{z}'_3 - \hat{z}'_2)^2 \quad (107)$$

describes a three-boson harmonic oscillator.

To handle Eq. (104) it is tempting to neglect its last term. Invoking the limit of a large total momentum ( $M^2 \rightarrow \infty$ ), as done in Droz-Vincent (2002)<sup>3</sup>, does not seem to permit a perturbation treatment (indeed the “small” parameter used in the developments should be independent from the eigenvalue). We prefer to consider developments in powers of  $1/c$ .

### 5.1. Equal Masses

Assuming for simplicity that  $m_a = m$ , Eq. (104) becomes

$$(9m^2 - M^2)c^2\psi = \underline{D}\psi + 18\underline{V}\psi + \frac{6}{M^2c^2}\Gamma\psi \quad (108)$$

It will be considered as an eigenvalue problem for  $\lambda$  by setting  $6\lambda = (M^2 - 9m^2)c^2$ . As all masses are equal, thus  $\Gamma = \Gamma(0) + O(1/c^2)$ . At first order in  $1/c^2$  we can replace  $\Gamma$  by  $\Gamma(0)$  and  $M^2$  by  $9m^2$  in (108). Using the rest frame, we obtain

$$\lambda\psi = [\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3 - 3\underline{V}]\psi - \frac{1}{9m^2c^2}\Gamma(0)\psi \quad (109)$$

Neglecting the last term yields the nonrelativistic limit (divide by  $m$  and remember that in our formulas,  $V$  has dimension of  $P^2$ ).

Taking into account the contribution of  $\Gamma(0)$  permits to calculate the first relativistic correction.

### 5.2. Cluster Behavior

As pointed out by Sazdjian (1988, 1989) in any formulation of the dynamics which makes explicit reference to total momentum, it is difficult to discuss cluster separability. But it is reasonable to demand that the *reduced* equation be in a sense separable, in order to ensure a factorization of the internal wave function when there are noninteracting clusters.

With this requirement in mind, we can already observe that the potential (105) is formally separable *in terms of the variables*  $z'$ .

But the interpretation of each  $U_{ab}$  as a two-body term runs into a complication: there is no evidence that the variable  $z'_A$  exactly matches the cluster of particles  $\{1A\}$ . A similar remark arises concerning the matching of  $z'_3 - z'_2$  with cluster  $\{23\}$ .

The physical interpretation of the new configuration variables  $z'_A$  is not straightforward; they are relative variables since they commute with  $P$ , but they suffer from this complication that the transformation formulas (25) mix  $z_2$  with  $z_3$ . Similarly (besides its dependence on total momentum) each new variable  $q'_a$  depends not only on  $q_a$  (with the same label  $a$ ) but also on all  $q_b$ 's with  $b \neq a$ .

<sup>3</sup> Several coefficients in the expression of  $\Gamma(0)$  written in that article must be corrected, see eq. (102) in the present paper.

But we can consider (91) on the momentum–mass shell. At least for three equal masses, the only occurrence of the velocity of light is through the product  $Mc$ , so we obtain from (91)

$$\hat{z}'_2 = \hat{z}_2 + O(1/|Mc|), \quad \hat{z}'_3 = \hat{z}_3 + O(1/|Mc|) \quad (110)$$

and, of course,  $\hat{z}'_3 - \hat{z}'_2 = \hat{z}_3 - \hat{z}_2 + O(1/|Mc|)$ . Thus, at leading order, the variables  $\hat{z}'_A$  and  $\hat{z}_A$  still coincide; so the potentials  $U_{ab}$  in the reduced Eq. (106) can be approximately considered as two-body terms.

## 6. CONCLUDING REMARKS

As a first step, we succeeded in constructing three mass-shell constraints describing the *free motion* of three scalar particles. In contrast to the KG system, these new wave equations permit to eliminate two degrees of freedom and get reduced to a covariant equation with three-dimensional arguments.

Our approach rests on a transformation of the momenta involved in the original KG system. In contradistinction to Sazdjian's proposal and the homographic relations that approximate it (Eq. (13) of Sazdjian, 1989, Eq. (4.15) of Sazdjian, 1988), our transformation from the old momenta to the new ones is explicitly given by simple *quadratic* formulas. Inversion of these formulas is a fourth-degree algebraic problem which could be (in principle) discussed and solved in closed form; due to its complexity, approximate developments are more efficient in practical calculations. We used a couple of identities that are specific of the three-body case; thus an extension of the present work to  $n > 3$  is by no means straightforward.

In the present state of the art, equivalence of the new Eq. (97) with the sum of the original KG equations is ensured *at least* in a large sector characterized by positive energies and conditions that involve the masses of the particles. When the masses are not too different one from another (and in particular for equal masses), these conditions amount to impose a bound on the velocities; but this bound is still high enough to allow for the description of a relativistic regime.

The case of very large velocities requires further investigations. We gave here *sufficient* conditions for an invertible transformation; it remains possible that a more detailed discussion enlarges the present results. This analysis of free-body kinematics provides us with a solid ground.

In a second step, we introduced *interaction* in the “sum equation.” The model obtained by this procedure respects Poincaré invariance. It remains covariantly reducible to a wave equation with three-dimensional arguments; free motion is recovered in the absence of interaction term.

The interaction term  $V$  is *formally* cluster separable; actually formula (105) is an ansatz which permits to combine two-body interactions without spoiling the compatibility of the mass-shell constraints. True separability (in terms of the original individual particle coordinates) is recovered only in the large-total-mass limit.

The two-body input of our model can be either phenomenological or motivated by consideration of field theory.

When the three masses are equal, the velocity of light arises in  $\Xi$  through the product  $M^2c^2$ , which facilitates the expansion in powers of  $1/c^2$ . At the first order, the reduced equation is similar to a familiar Schroedinger equation supplemented with a perturbation; insofar as the interaction is not explicitly energy-dependent (or if this dependence is of higher order) one is left with a conventional eigenvalue problem.

This situation provides a basis for eventually undertaking the study of cases where the mass differences are not zero but still remain relatively small.

In the hope of applications to realistic bound states, like three-quark systems, we plan an extension of the formalism to particles with spin. The contact with more elaborated (but more complicated) theories, such as QED and QCD, will be discussed in a future work.

## APPENDIX A

In non-relativistic classical mechanics, canonical transformations are symplectic diffeomorphisms of phase space. In general they do mix  $q$ 's and  $p$ 's.

But a *point transformation* (in configuration space) simply transforms  $q$ 's among themselves, say  $q' = f(q)$ . Then invariance of the symplectic form (Goldstein, 1964) fully determines  $p$ 's in terms of  $q$ 's and  $p$ 's. When configuration space is flat, the  $q$  and  $p$  variables play symmetric roles in the general formulas of analytic mechanics, so there is no difficulty in defining as well point transformations in momentum space (but this possibility is not usually considered in textbooks). In this case, one transforms the momenta among themselves, and one further determines the new variables  $q'$  in terms of  $q$ 's and  $p$ 's through the requirement that the complete transformation law is canonical.

In the position (resp. momentum) representation of quantum mechanics, a quantum analog of point transformations in configuration (resp. momentum) space can be generated by an invertible transformation of the arguments of the wave function. This transformation among  $c$ -numbers obviously induces a transformation among the multiplicative operators they define.

## APPENDIX B

The polynomial  $R_\sigma(X)$  has an obvious double root  $X = 2/3$  independent of  $\sigma$ , and provided  $\sigma > -1$ , two other real roots given by (59) but, as noticed above, the root  $X^-$  falls outside the admissible interval. All the curves  $Z = R_\sigma(X)$  are tangent to the  $X$  axis at a fixed point  $X = 2/3$ . For  $\sigma > -1$  and  $\epsilon$  small enough, the curve representing  $R(X)$  is four times cut by the straight line  $Z = \epsilon^2$ . In the limit when  $\epsilon$  vanishes, two points of this intersection form the contact with the  $X$  axis, and the other ones respectively reduce to  $X^+$  and  $X^-$ .



For  $\sigma = \frac{1}{2}$  we find that  $X^+ \simeq 0.15$ , which is admissible in the sense of (45), (46), and  $R_{1/2}(X)$  has a local maximum at  $X = 1/3$ . This maximum is exactly  $\frac{1}{4}$ . For  $\sigma < \frac{1}{2}$ , the local maximum exceeds  $\frac{1}{4}$ . Making  $\sigma$  to decrease, we obtain lower values of  $X^+$  (which vanishes with  $\sigma$ ).

For  $\sigma = -\frac{3}{4}$ , we obtain exactly  $X^+ = -\frac{1}{3}$ , and going down further is excluded in view of (46).

Taking  $\sigma$  in the open interval  $(-\frac{3}{4}, \frac{1}{2})$ , and  $X$  restricted by  $-\frac{1}{3} < X < \frac{2}{3}$ , it turns out that, *provided*  $\epsilon^2 < \frac{1}{4}$ , each curve  $Z = R_\sigma(X)$  has two points in common with the straight line  $Z = \epsilon^2$  (other possible points correspond to  $X$  outside the interval we consider). For vanishing  $\epsilon$ , one of them has its horizontal coordinate going to coincide with  $X^+$ , while the other point goes to the fixed contact point  $X = 2/3$ ,  $Z = 0$ . This analysis shows that, with our restrictions, the fourth-degree equation  $R_\sigma(X) = \epsilon^2$  has two real solutions, but only one of them reduces to  $X^+$  in the limit where  $\epsilon$  vanishes.

## REFERENCES

- Basdevant, T. L. and Omnès, R. (1966). *Physical Review Letters* **17**, 775.
- Bel, L. (1976). In "Differential Geometry and Relativity," M. Cahen and M. Flato eds., Reidel, Dordrecht p. 197.
- Bel, L. (1983). *Physical Review D: Particles and Fields* **28**, 1308.
- Bijtebier, J. (1990). Three-fermion relativistic bound states and cluster decomposition. *Nuovo Cimento A* **103**, 669–682.
- Bijtebier, J. (1999). Two and three-fermion 3D equations deduced from Bethe-Salpeter equations. *Preprints hep-th/9912099*.
- Bijtebier, J. (2000). *Journal Physics G* **26**, 871–886.
- Bijtebier, J. (2001). *Nuclear Physics A* **696**, 581.
- Crater, H. W., and Van Alstine, P. (1983). *Annals of Physics (New York)* **148**, 57.
- Crater, H. W., and Van Alstine, P. (1987). *Physical Review D: Particles and Fields* **36**, 3007.
- Crater, H. W., Becker, R. L., Wong, C. Y., and Van Alstine, P. (1992). *Physical Review D: Particles and Fields* **46**, 5117.
- Droz-Vincent, Ph. (1975). *Reports in Math. Phys.* **8**, 79.
- Droz-Vincent, Ph. (1979). *Physical Review D: Particles and Fields* **19**, 702.
- Droz-Vincent, Ph. (1985). *Physics Letters B* **159**, 393.
- Droz-Vincent, Ph. (2002). *Few-Body Systems* **31**, 165–170.
- Goldstein, H. (1964). *Classical Mechanics*, chap. 8, section 8–2, Addison-Wesley, Reading, MA.
- Iranzo, V., Llosa, J., Marqués, F., Molina, A. (1984). *Annals de l'Institut Henri Poincaré Section A: Physique Théorique* **35**, 1; **40**, 1.
- Leutwyler, H. and Stern, J. (1978a). *Annals of Physics (New York)* **112**, 94.
- Leutwyler, H. and Stern, J. (1978b). *Physics Letters B* **73**, 75.
- Llosa, J. ed. (1982). *Relativistic Action-at-a-distance Classical and Quantum Aspects Vol. 162*, Lecture Notes in Physics Springer-Verlag, and references therein.
- Pagnamenta, A. (1996a). In *Lectures in Theoretical Physics*, University of Colorado, Boulder.
- Pagnamenta, A. (1969b). *Mathematical Methods in Theoretical Physics*, Gordon and Breach, New York.
- Rizov, V. A., Sazdjian, H., and Todorov, I. T. (1985). *Annals of Physics* **165**, 59.

- Rupp, G. and Tjon, J. A. (1992). *Physical Review C: Nuclear Physics* **45**, 2133.
- Sammarruca, F., Xu, D.P., and Machleidt, R. (1992). *Physical Review C: Nuclear Physics* **46**, 1636.
- Sazdjian, H. (1985). *Physics Letters B* **156**, 381.
- Sazdjian, H. (1987). *Journal Mathematical Physics* **28**, 2618.
- Sazdjian, H. (1988). Relativistic dynamics for N-body systems, *Physics Letters B* **208**, 470–474.
- Sazdjian, H. (1989). N-Body bound state relativistic wave equations. *Annals of Physics (New York)* **191**, 52–77.
- Stadler, A. and Gross, F. (1997). *Physical Review Letters* **78**, 26.
- Taylor, J. G. (1966). *Physical Review* **150**, 1321.
- Todorov, I. T. (1976). JINR Report E2-10125. Unpublished.